An Axiomatic Characterization of Lin’s type Information Measure

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ABSTRACT

A new information measure \( I_n(P/Q/R) \) of Lin’s type for three probability distributions is introduced. Its axiomatic characterization is on the basis of some axiom. The Information measure satisfies most of the desirable properties of Information measure \( I_n \). A theorem for this measure is proved with the help of some lemma’s.

1. Introduction :-

Let,

\[
P = (p_1, p_2, p_3, \ldots, p_n), \quad p_i \geq 0, \quad \sum_{i=1}^{n} p_i = 1
\]

\[
Q = (q_1, q_2, q_3, \ldots, q_n), \quad q_i \geq 0, \quad \sum_{i=1}^{n} q_i = 1
\]

and \( R = (r_1, r_2, r_3, \ldots, r_n), \quad r_i \geq 0, \quad \sum_{i=1}^{n} r_i = 1
\)

be the actual, predicted and revised probability distributions associated with the events \( E = (E_1, E_2, E_3, \ldots, E_n) \) on the basis of an experiment respectively. Theil (1967) defined information improvement as

\[
I(P/Q/R) = \sum_{i=1}^{n} p_i \log \frac{r_i}{q_i}
\]

Lin (1991) introduces a new directed divergence, which overcome the difficulty of absolute continuity. Lin’s divergence measure denoted by \( K(P/Q) \) between two probability distribution \( P \) and \( Q \) is defined as

\[
K(P/Q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{p_i + q_i}
\]

The new information measure for three probability distributions can be defined as

\[
I_n(P/Q/R) = \sum_{i=1}^{n} p_i \log \frac{2r_i}{r_i + q_i}
\]

The generalised minimum cross-entropy principal (GMCEP) states that

1. A posterior probability distribution \( P(p_1, p_2, p_3, \ldots, p_n) \)
2. Set \( C \) of moment constraints of the form

\[
\sum_{i=1}^{n} p_i = 1, \quad \sum_{i=1}^{n} p_i g_r(x_i) = a_r, \quad r = 1, 2, 3, \ldots, m
\]

3. A priori probability distribution \( Q(q_1, q_2, q_3, \ldots, q_n) \)
(4) A measure of directed divergence or cross entropy
\[ D(P; Q) = D(p_1, p_2, p_3, \ldots, p_n, q_1, q_2, q_3, \ldots, q_n) \]
Satisfying the following conditions:
(a) \[ D(P; Q) \geq 0, \]
(b) \[ D(P; Q) = 0 \text{ iff } P = Q \text{ or } p_i = q_i \text{ for each } i \]
(c) \[ D(P; Q) \text{ is a convex function of } p_1, p_2, p_3, \ldots, p_n \]
(d) \[ \text{when } D(P; Q) \text{ is minimized subject to } C, \text{ the minimizing probabilities are all } \geq 0 \]

2. Characterization of \( I_n(P/Q/R) \):
For characterizing the measure \( I_n(P/Q/R) \), we assume the following axioms.

**Axiom I (Continuity):**
\[ I_n(P/Q/R) = \sum_{i=1}^{n} p_i \log \frac{2r_i}{r_i + q_i} \text{is a continous function of its arguments.} \]

**Axiom II (Symmetry):** The measure \( I_n(P/Q/R) \) is a symmetric function of its arguments, that is
\[ I_n(p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n, r_1, r_2, \ldots, r_n) = \]
\[ I_n(p_{k(1)}, p_{k(2)}, \ldots, p_{k(n)}, q_{k(1)}, q_{k(2)}, \ldots, q_{k(n)}, r_{k(1)}, r_{k(2)}, \ldots, r_{k(n)}) \]
Where \( k \) is an arbitrary permutation on \((1, 2, 3, \ldots, n)\).

**Axiom III (Expansibility):**
\[ I_{n+1}(p_1, \ldots, p_n; q_1, \ldots, q_n; r_1, \ldots, r_n) = I_{n+1}(0; p_1, \ldots, p_n; q_1, \ldots, q_n; r_1, \ldots, r_n) \]
\[ = I_{n+1}(p_1, \ldots, p_n; 0, q_1, \ldots, q_n; r_1, \ldots, r_n) \]
\[ = I_{n+1}(p_1, \ldots, p_n; q_1, \ldots, q_n; 0, r_1, \ldots, r_n) \]
\[ = I_{n+1}(p_1, \ldots, p_n; q_1, \ldots, q_n; r_1, \ldots, r_n; 0) \]
... \[ = I_{n+1}(p_1, \ldots, p_n; q_1, \ldots, q_n; r_1, \ldots, r_n; 0) \]

**Axiom IV (Branching property):**
\[ I_{n+1}(p_1, p_2, \ldots, p_{n-1}, p', p'', q_1, q_2, \ldots, q_{n-1}, q', q'', r_1, r_2, \ldots, r_{n-1}, r', r'') = \]
\[ = I_n(P/Q/R) - p_n r_n q_n + p' q' r' + p'' q'' r'' \]
Proof:- Taking right hand side,
\[ = I_n(P/Q/R) - p_n r_n q_n + p' q' r' + p'' q'' r'' \]
\[ = \sum_{i=1}^{n-1} p_i \log \frac{2r_i}{r_i + q_i} + p_n r_n q_n - p_n \log \frac{2r_n}{r_n + q_n} + p' \log \frac{2r'}{r' + q'} + p'' \log \frac{2r''}{r'' + q''} \]
\[ = I_{n+1}(p_1, p_2, \ldots, p_{n-1}, p', p'', q_1, q_2, \ldots, q_{n-1}, q', q'', r_1, r_2, \ldots, r_{n-1}, r', r'') \]
\[ = \text{L.H.S.} \]

**Axiom V:**
\[ I_n \left( \frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right) = \log \left( \frac{2b}{b + s} \right) = a(2b, b + s) \]
Proof:- \[ \text{L.H.S.}= \]
\[ \frac{1}{n} \log \left( \frac{2 \times \frac{1}{s}}{\frac{1}{s} + \frac{1}{b}} \right) + \ldots + \frac{1}{n} \log \left( \frac{2 \times \frac{1}{s}}{\frac{1}{s} + \frac{1}{b}} \right) \]
= \frac{1}{n} \log \left( \frac{2b}{b + s} \right) + \cdots + \frac{1}{n} \log \left( \frac{2b}{b + s} \right)
\text{Axiom VI:-} \quad l_2 (1, 0; 1, 0; 1, 0) = 0

3. Theorem:- \quad \text{If } P, Q, R \text{ be the actual, predicted and revised probability distributions associated with the events such that } \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = \sum_{i=1}^{n} r_i = 1, \text{ then information measure for three probability distribution can be defined as}

\[ I_n (P/Q/R) = \sum_{i=1}^{n} p_i \log \frac{2r_i}{r_i + q_i} \]

Proof:- Before the proof of the theorem, we will prove some lemma’s.

Lemma (i):-

If \( v_k \geq 0, \sum_{k=1}^{n} v_k = p_i > 0, t_k \geq 0, \sum_{k=1}^{n} t_k = q_i > 0, w_k \geq 0, \sum_{k=1}^{n} w_k = r_i > 0, \text{ then}

\[ I_{m+n+1} (p_1, p_2, \ldots, p_{m-1}, v_1, v_2 \ldots, v_m, p_i+1, \ldots, p_n; q_1, q_2, \ldots, q_{m-1}, t_1, t_2 \ldots, t_m, q_{i+1} \ldots, q_n, r_1, r_2, \ldots, r_{m-1}, w_1, w_2 \ldots, w_m, r_i+1, \ldots, r_n) \]

\[ = I_n (P/Q/R) + \sum_{j=1}^{m} v_j \log \frac{2w_j}{w_j + t_j} - p_i \log \frac{2r_i}{r_i + q_i} \]

Proof of Lemma:-

\[ I_n (P/Q/R) + \sum_{j=1}^{m} v_j \log \frac{2w_j}{w_j + t_j} - p_i \log \frac{2r_i}{r_i + q_i} = \]

\[ = \sum_{i=1}^{n} p_i \log \frac{2r_i}{r_i + q_i} + p_i \log \frac{2r_i}{r_i + q_i} + \sum_{j=1}^{m} v_j \log \frac{2w_j}{w_j + t_j} - p_i \log \frac{2r_i}{r_i + q_i} \]

\[ = I_{m+n+1} (p_1, p_2, \ldots, p_{m-1}, v_1, v_2 \ldots, v_m, p_i+1, \ldots, p_n; q_1, q_2, \ldots, q_{m-1}, t_1, t_2 \ldots, t_m, q_{i+1} \ldots, q_n, r_1, r_2, \ldots, r_{m-1}, w_1, w_2 \ldots, w_m, r_i+1, \ldots, r_n) \]

Lemma (ii):-

If \( v_{ij} \geq 0, j = 1, 2, \ldots, m, i = 1, 2, \ldots, n, \quad \sum_{j=1}^{m} v_{ij} = p_i > 0, t_{ij} \geq 0, \sum_{j=1}^{m} t_{ij} = q_i > 0; \)

\[ w_{ij} \geq 0, \sum_{j=1}^{m} w_{ij} = r_i > 0, \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = \sum_{i=1}^{n} r_i, \text{ then}

\[ I_{m_1+m_2+\ldots+m_n} (v_{11}, \ldots, v_{1m_1}, \ldots, v_{n1}, \ldots, v_{nm_1}, t_{11}, \ldots, t_{1m_1}, \ldots, t_{n1}, \ldots, t_{nm_1}; r_{11}, \ldots, r_{1m_1}, \ldots, r_{nm_1}) = \]

\[ = I_n (P/Q/R) + \sum_{i=1}^{n} \sum_{j=1}^{m_i} v_{ij} \log \frac{2w_{ij}}{w_{ij} + t_{ij}} - \sum_{i=1}^{n} p_i \log \frac{2r_i}{r_i + q_i} \]

Proof of lemma:-

\[ = I_n (P/Q/R) + \sum_{i=1}^{n} \sum_{j=1}^{m_i} v_{ij} \log \frac{2w_{ij}}{w_{ij} + t_{ij}} - \sum_{i=1}^{n} p_i \log \frac{2r_i}{r_i + q_i} \]
= \sum_{i=1}^{n} p_i \log \frac{2r_i}{r_i + q_i} + \sum_{i=1}^{n} \sum_{j=1}^{m_i} v_{ij} \log \frac{2w_{ij}}{w_{ij} + t_{ij}} - \sum_{i=1}^{n} p_i \log \frac{2r_i}{r_i + q_i}

= \sum_{i=1}^{n} \sum_{j=1}^{m_i} v_{ij} \log \frac{2w_{ij}}{w_{ij} + t_{ij}}

= \sum_{i=1}^{n} \left[ v_{i1} \log \frac{2w_{1i}}{w_{1i} + t_{11}} + v_{i2} \log \frac{2w_{2i}}{w_{2i} + t_{12}} + \ldots + v_{im_i} \log \frac{2w_{m_i}}{w_{m_i} + t_{1m_i}} \right]

= \left( v_{11} \log \frac{2w_{11}}{w_{11} + t_{11}} + v_{21} \log \frac{2w_{21}}{w_{21} + t_{12}} + \ldots + v_{n1} \log \frac{2w_{n1}}{w_{n1} + t_{1n}} + \ldots \right)

+ \sum_{k=1}^{n} v_{km_n} \log \frac{2w_{km_n}}{w_{km_n} + t_{km_n}}

= I_{m_1+m_2+\ldots+m_n} (v_{11}, \ldots, v_{1m_1}, \ldots, v_{nm_n}, t_{11}, \ldots, t_{1m_1}, \ldots, t_{n1}, \ldots, t_{nm_n}; w_{11}, \ldots, w_{1m_1}, \ldots, w_{n1}, \ldots, w_{nm_n})

= L.H.S.

Lemma (iii):

\[ a(2b, b + s) = \log \frac{2b}{s + b} \]

Proof of lemma:- in lemma (ii), replace \( m_i \) by

\[ m \] for each \( i \) and \( v_{ij} = \frac{1}{mn}, t_{ij} = \frac{1}{b_1 b_2}, q_i = \frac{1}{b_2} \]

\[ w_{ij} = \frac{1}{s_1 s_2}, r_i = \frac{1}{s_2}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, m \] and put \( b_2 = s_2 = 1 \)

\[ I_{mn} \left( \frac{1}{mn}, \ldots, \frac{1}{mn}, \frac{1}{b_1 b_2}, \frac{1}{b_1 b_2}, \ldots, \frac{1}{b_1 b_2}, \frac{1}{s_1 s_2}, \frac{1}{s_1 s_2}, \ldots, \frac{1}{s_1 s_2} \right) = \]

\[ I_{n}(P/Q/R) + \frac{1}{mn} \log \frac{2b_1 b_2}{b_1 b_2 + s_1 s_2} + \ldots + \frac{1}{mn} \log \frac{2b_1 b_2}{b_1 b_2 + s_1 s_2} - \sum_{i=1}^{n} p_i \log \frac{r_i}{q_i} \]

By Axiom IV,

\[ a(2b_1 b_2, b_1 b_2 + s_1 s_2) = \frac{1}{mn} (1 + 1 + \ldots + 1) \log \frac{2b_1 b_2}{s_1 s_2 + b_1 b_2} \]

\[ \Rightarrow a(2b_1, b_1 + s_1) = \log \frac{2b_1}{s_1 + b_1} \]

\[ \Rightarrow a(2b, b + s) = \log \frac{2b}{s + b} \]

Proof of theorem:- If \( m_i, a_i, b_i \) and \( s_i \) are positive integer such that

\[ \sum_{i=1}^{n} a_i = m, \quad a_i \leq b_i, \quad i = 1, 2, \ldots, n \]

\[ \text{and} \quad p_i = \frac{a_i}{m}, q_i = \frac{b_i}{b}, r_i = \frac{s_i}{s}, \text{where} \sum_{i=1}^{n} b_i = b, \sum_{i=1}^{n} s_i = s \]

Then by application of lemma (ii) i.e.
\[
I_m \left( \frac{1}{m}, \frac{1}{m}; \frac{1}{b}, \frac{1}{b}; \ldots; \frac{1}{s}, \frac{1}{s}; \frac{1}{s}, \frac{1}{s} \right) = I_n(P/Q/R) + \sum_{i=1}^{n} \sum_{j=1}^{a_i} \frac{1}{m} \log \frac{2b}{s+b} + \cdots + \frac{1}{s} \log \frac{2b}{s+b} + \cdots \\
+ \sum_{i=1}^{n} \sum_{j=1}^{a_i} \frac{1}{m} \log \frac{2}{s+b} - \sum_{i=1}^{n} p_i \log \frac{2r_i}{r_i + q_i}
\]

\[
\Rightarrow I_n(P/Q/R) = - \sum_{i=1}^{n} \sum_{j=1}^{a_i} \frac{m}{m} \log \frac{2b}{s+b} + \sum_{i=1}^{n} p_i \log \frac{2r_i}{r_i + q_i} + a(2b, b + s)
\]

Using Axiom V,

\[
I_n(P/Q/R) = - \log \frac{2b}{s+b} + \sum_{i=1}^{n} p_i \log \frac{2r_i}{r_i + q_i} + a(2b, b + s)
\]

Using lemma (iii),

\[
I_n(P/Q/R) = -a(2b, b + s) + \sum_{i=1}^{n} p_i \log \frac{2r_i}{r_i + q_i} + a(2b, b + s)
\]

\[
I_n(P/Q/R) = \sum_{i=1}^{n} p_i \log \frac{2r_i}{r_i + q_i}
\]

4. Conclusions:

Thus author has introduced a new measure based on axioms that can overcome the previous difficulties. Author have also tried to show that the measure satisfy most of the desirable properties of Information measure \( I_n \).

References