Double Domination Number of Butterfly Graphs BF(n)

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ABSTRACT
A family of graphs called Butterfly Graphs is introduced recently and study of its parameters is under progress. Butterfly Graphs are widely used in interconnection networks. In this paper, we find double dominating sets of Butterfly graph and show that double domination number of BF(n) is \( \gamma_{dd}(BF(n)) = n^{2n-1} \).

Keywords: Butterfly Graphs, Dominating set, double domination number,

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INTRODUCTION
A subset D of V(G) is called a double dominating set of G if every vertex in V is dominated by at least two vertices in D. Cardinality of the minimum double dominating set is called the double domination number of G and is denoted by \( \gamma_{dd}(G) \).

In this paper we present double domination number of butterfly graphs BF(n). First we give construction of a double dominating set for BF(n), n=2, 3 The values for \( n > 3 \) are dealt using Recursive Construction 2 for BF(n).

MAIN RESULTS
Lemma 1. : Double domination number of BF(2) is 4.
Proof : Consider the graph BF(2). \( \gamma(BF(2)) = 2[7] \). As BF(2) is a 2-partite graph for two domination of vertices in level L_0 we have to include two vertices from L_1 and vice versa. Therefore cardinality of a double dominating set >=4. Consider D = \{ (0;0), (0;1), (1;2), (1;3) \}. We claim that D is a double dominating set of BF(2). First \{ (0;0) (1;2) \} and \{ (0;1) (1;3) \} are pairs of adjacent vertices in D and so they double dominate each other. Consider the vertex (1;0). It is adjacent to two vertices (0;0), (0;1) in D by straight edge and winged edge respectively. Similarly is the case with the vertex (1;1). Consider the vertex (0;2). It is adjacent with (1;2) and (1;3) by straight and winged edges respectively. Similar is the case with the vertex (0;3). This shows that all the vertices in V are double dominated by the vertices in D. Therefore D is a double dominating set of minimum cardinality 4. Hence \( \gamma_{dd}(BF(2)) = 4 \).

Lemma 2 : The double domination number of BF(3) is 12.
Proof : Consider the graph BF(3). BF(3) is the only butterfly graph with triangles[5]. It has 8 disjoint triangles with one vertex each from L_0, L_1, L_2. Two vertices from a triangle double dominate the third vertex. Suppose we select two vertices from each triangle from levels L_0 and L_1 into D, this set of 16 vertices is a double dominating set of BF(3). So \( \gamma_{dd}(BF(3)) \leq 16 \). Next, all 8 vertices from any one level will double dominate all vertices of remaining 2 levels. So all vertices of L_0 and L_2 are double dominated by vertices of L_1 in D by straight and slanting edges. To double dominate vertices of L_1, we include 4 vertices of L_0 (or L_2) such that each of these dominates
two distinct vertices of \(L_1\). Thus \(D\) become a double dominating set of \(BF(3)\) of cardinality 12. \(D\) is a minimum double dominating set of \(BF(3)\). Hence \(\gamma_{dd}(BF(3)) = 12\).

**Theorem 3**: The double domination number of \( BF(3k) \) is \( 3k \times 2^{3k-1} \).

**Proof**: We prove the result by using the Principle of Mathematical Induction on \(k\).

**Step 1**: Let \(k = 1\). Then \(n = 3\). From Lemma 2 the result is true for \(BF(3)\).

**Step 2**: Let us assume that the result is true for \(k = t\). We prove the result for \(k = t+1\). Consider the graph \(BF(3(t+1))\). From Recursive Construction 2 [ ], \(BF(3t+3)\) is decomposed into 8 copies of \(BF(3t)\) without wings and the last 3 levels form a pattern of \(BF(3)\) with \(2^3\) vertex groups, where each vertex group has \(2^3\) vertices. From the induction hypothesis the result is true for \(BF(3t)\).

Consider a double dominating set \(S_i\) of \(BF(3t+3)\) for \(i = 1, 2, 3\ldots 8\) copies of \(BF(3t)\). The last 3 levels of \(BF(3t+3)\) form a pattern of \(BF(3)\) which is isomorphic to \(BF(3)\) without wings. Using Lemma 2, we get a double dominating set \(T_1\) for this pattern of \(BF(3)\) with 8 vertex groups, where each vertex group has \(2^3\) vertices.

Suppose we get a double dominating set of \(BF(3)\) of cardinality less than \(2^3 \times (3t 2^{3t-1})\) for \(BF(3)\). Since pattern of \(BF(3)\) is isomorphic to the graph \(BF(3)\) without wings, this assumption gives that we can get a double dominating set for \(BF(3)\) of cardinality less than 12, which is a contradiction as \(\gamma_{dd}(BF(3)) = 12\).

Let \(D = g D_i S_i \cup T_i\)

For \(i = 1, 2 \ldots 8\), each \(S_i\) is a minimum double dominating set of \(BF(3t)\) between levels \(L_0\) to \(L_{3t+1}\)

and \(T_1\) is a minimum double dominating set of \(BF(3t)\) between \(L_{3t}\) to \(L_{3t+2}\). As all the \(S_i\)'s and \(T_1\) are disjoint sets \(D\) is a minimum double dominating set of \(BF(3t+3)\) of cardinality

\[
|D| = 8 \times 3 \times 3 \times 2^{3t-1} + 8 \times 3 \times 2^{3t-1} = 3 \times (t+1) \times 2^{3t+2} = 3 \times (t+1) \times 2^{3(t+1)-1}
\]

So the result is true for \(k = t+1\). Hence by the Principle of Mathematical Induction the result is true for all positive integers \(k\). Thus \(\gamma_{dd}(BF(3k)) = 3k \times 2^{3k-1}\).

**Corollary 4**: The double domination number of \(BF(3k+r)\) is \(3k+r \times 2^{3k+r-1}\) for \(r = 1, 2\).

**Proof**: By Recursive Construction 1[ ], there are \(2^r\) copies of \(BF(3k)\) and \(r\) levels with \(2^{3k+r}\) vertices in \(BF(3k+r)\). Suppose \(D_r\) are double dominating sets for \(2^r\) copies of \(BF(3k)\) which double dominate all vertices from levels \(L_0\), \(L_1\), \ldots \(L_{3k-1}\). Let \(D = \overset{2^r}{\underset{i=1}{\bigcup}} D_i\).

Case (i) for \(r=1\). The vertices of \(L_{3k-1}\) dominate vertices of \(L_{3k}\), for double domination of these vertices we include all \(2^{3k+1}\) vertices of \(L_{3k-1}\) into \(D\), all the vertices of \(BF(3k+1)\) are double dominated giving \(D\) as a minimum double dominating set of \(BF(3k+1)\).

\[
D = \overset{2^r}{\underset{i=1}{\bigcup}} D_i \cup V(L_{3k-1})
\]

Hence \(\gamma_{dd}(BF(3k+1)) = 2 \times \gamma_{dd}(BF(3k)) + 2^{3k+1} = 2 \times 3k \times 2^{3k+1} + 2^{3k} = (3k+1) 2^{3k}\).

Case (ii) for \(r=2\). Now select all the vertices of \(L_{3k}\) into \(D\), these vertices double dominate all the vertices of \(L_{3k+1}\). Thus all vertices of \(BF(3k+2)\) are double dominated. As there is no choice other than selecting the vertices of \(L_{3k}\) to double dominate themselves and also vertices of \(L_{3k+1}\), it follows that this choice is minimum.
Hence $\gamma_{dd}(BF(3k+2)) = 2^2 \times \gamma_{dd}(BF(3k)) + 2^{3k+2} = 2^2 \times 3k \times 2^{3k-1} + 2^{3k+2} = (3k+2) 2^{3k+1}$. □

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